For electron form, introduce Harned-Mikic terms

$$
\begin{aligned}
\frac{1}{R^2}\dot{F} &= \nabla_\perp \bullet \left[-F\nabla_\perp U \times \nabla\varphi + \omega\nabla_\perp\psi \times \nabla\varphi - \frac{1}{R^4}F\nabla_\perp\chi - \omega\nabla_\perp f' \right] \\
&+ \nabla_\perp \bullet \eta \left[\frac{1}{R^2}\nabla F^* - \frac{1}{R^2}\nabla_\perp\psi' \times \nabla\varphi \right] \\
&+ d_i \nabla_\perp \bullet \left[\frac{1}{\rho R^2}\Delta^*\psi\nabla_\perp\psi \times \nabla\varphi + \frac{F}{\rho R^2}\nabla_\perp F^* \times \nabla\varphi + \frac{1}{\rho}\nabla_\perp p_e \times \nabla\varphi + {\color{red}\frac{F}{\rho R^4}\nabla_\perp\psi'} - \frac{1}{\rho R^2}\Delta^*\psi\nabla_\perp f' \right]
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\perp \bullet \frac{1}{R^2}\nabla\dot{\psi} &= \nabla_\perp \bullet \frac{1}{R^2}\nabla r^2 [U,\psi] - \nabla_\perp \bullet \frac{1}{R^2}\nabla r^2 (U, f') + \nabla_\perp \cdot \left[\frac{F}{R^2}\nabla_\perp U \right]' - \nabla_\perp \cdot \left[\frac{\omega}{R^2}\nabla_\perp\psi + \omega\nabla_\perp f' \times \nabla\varphi \right]' \\
&- \nabla_\perp \bullet \frac{1}{R^2}\nabla r^{-2} (\chi,\psi) - \nabla_\perp \bullet \frac{1}{R^2}\nabla R_0^2 [\chi, f'] - \nabla_\perp \cdot \left[\frac{F}{r^2 R^2}\nabla_\perp\chi \times \nabla\varphi \right]' \\
&+ \nabla_\perp \bullet \frac{1}{R^2}\nabla\eta\Delta^*\psi + \nabla_\perp \cdot \left[\frac{\eta}{R^2}\nabla F^* \times \nabla\varphi + \frac{\eta}{R^4}\nabla_\perp\psi' \right]' \\
&+ \nabla_\perp \bullet \frac{1}{R^2}\nabla\frac{d_i}{\rho}\left[\left[\psi, F^*\right] + \left(F^*, f'\right) + \frac{1}{R^2}(\psi,\psi') + [\psi', f'] + p_e' \right] \\
&- \nabla_\perp \cdot d_i \left[\begin{aligned} &\frac{1}{R^4\rho}\Delta^*\psi\nabla_\perp\psi {\color{red}+ \frac{F}{R^4\rho}\nabla_\perp F^*} + \frac{1}{R^2\rho}\nabla_\perp p_e \\ &- \frac{F}{R^4\rho}\nabla_\perp\psi' \times \nabla\varphi + \frac{1}{R^2\rho}\Delta^*\psi\nabla_\perp f' \times \nabla\varphi \end{aligned} \right]'
\end{aligned}
$$

$$(\;)' \equiv \frac{\partial}{\partial\varphi}(\;)$$

$$
\begin{aligned}
\mathbf{B} = \nabla{\color{red}\psi} \times \nabla\varphi \\
-\nabla_\perp f' + {\color{red}F}\nabla\varphi
\end{aligned}
$$

$$\nabla_\perp^2 f = F - F_0$$

Dominant cross terms in field advance are marked in red

Harned-Mikic terms -- 2

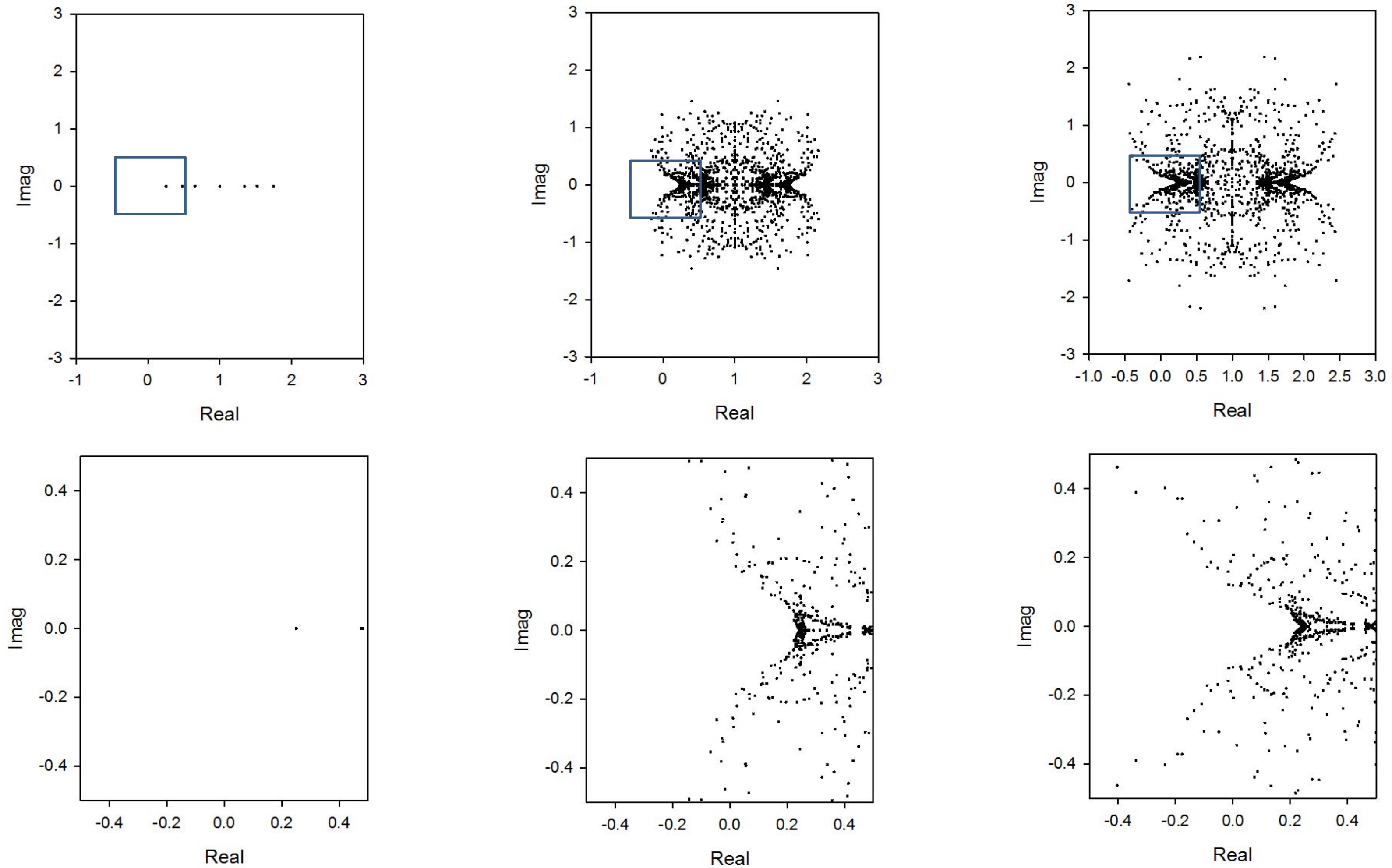
Implicit advance for the field variables have the Harned-Mikic terms added to make the matrix more diagonal and improve the 3D iterative solution when 2F terms are present.

$$\nabla_\perp \bullet \frac{1}{R^2}\nabla_\perp \dot{\psi} + \left(\theta\,\delta t\, d_i\right)^2 (H_m)\nabla_\perp \bullet \left[\underbrace{\frac{F}{R^4 n}\nabla_\perp \left(R^2 \nabla_\perp \bullet \frac{F}{R^4 n}\nabla_\perp \dot{\psi}'' \right)}_{\left(\mathbf{B}\bullet\nabla\right)^2 \nabla_\perp^2} \right] = \cdots$$

$$\frac{1}{R^2}\dot{F} + \left(\theta\,\delta t\, d_i\right)^2 (H_m)\underbrace{\frac{F}{nR^2}\nabla_\perp \bullet \frac{F}{nR^4}\nabla_\perp \dot{F}''}_{\left(\mathbf{B}\bullet\nabla\right)^2 \nabla_\perp^2} = \cdots$$

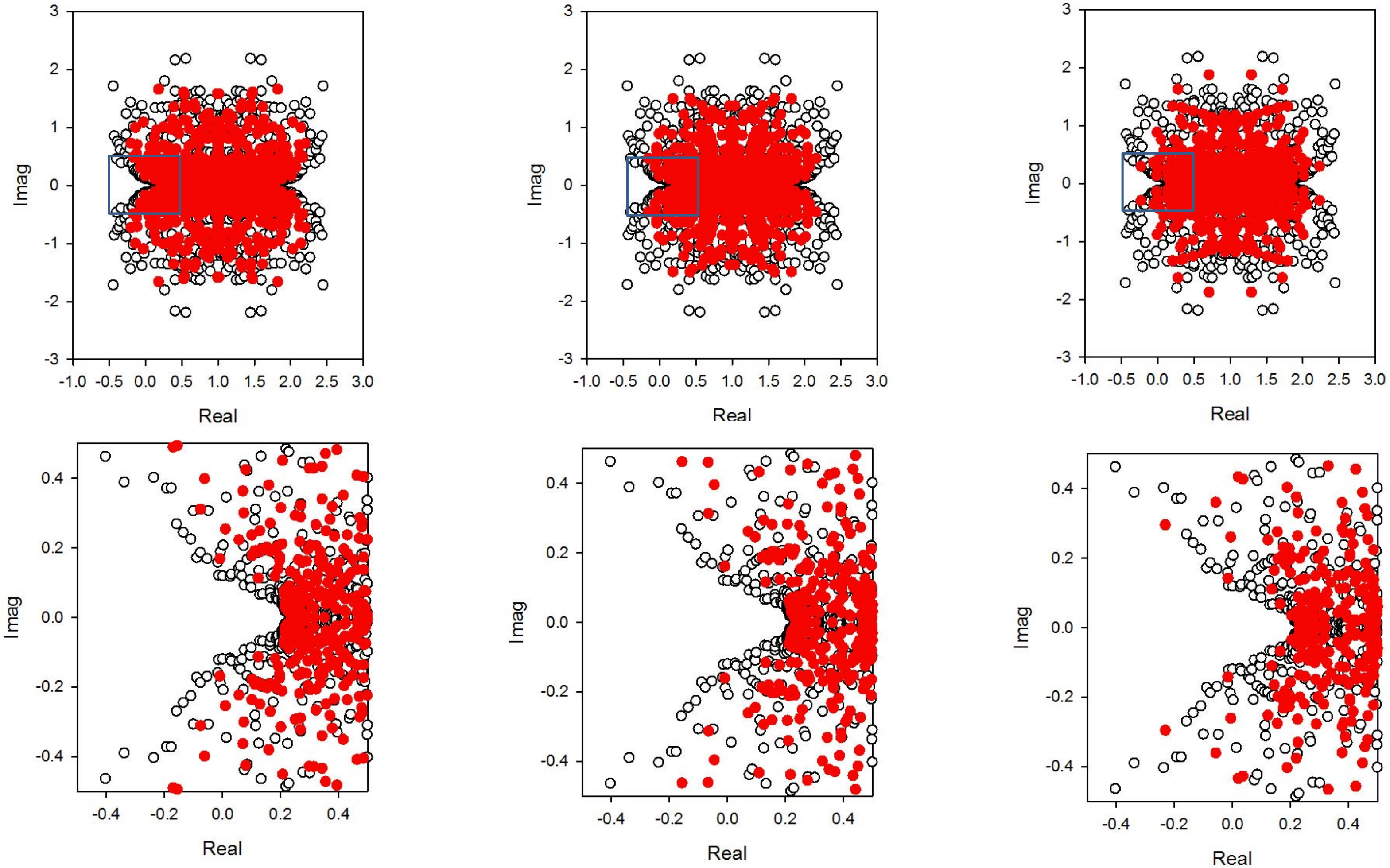
Eigenvalues of the field matrix after preconditioning (with no HM terms)

$d_i = 0$ $d_i = .045$ $d_i = .090$



Eigenvalues of the field matrix with d_i=.09 after preconditioning showing the effect of the Harned-Mikic terms (in red)

HM=.05 HM=.10 HM=.20



There is some improvement in the ratio of largest to smallest eigenvalues, but since matrix is non-symmetric, the significance of this is unclear.